The Gauss-Jordan method

Algorithm

The Gauss-Jordan method transforms the initial expanded matrix into a identity matrix and a right side equal to the solution that is sought:

$$(A\,|\,b)=\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ & \dots\dots & & & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right) \leftrightarrow \left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & c_1 \\ 0 & 1 & \dots & 0 & c_2 \\ & \dots\dots & & & \dots \\ 0 & 0 & \dots & 1 & c_n \end{array}\right) \leftrightarrow \boxed{\begin{array}{c} x_1 = c_1 \\ x_2 = c_2 \\ \dots\dots \\ x_n = c_n \end{array}}$$

Example 1. Solve the system using the Gauss-Jordan method with a chosen pivot element from a row:

$$\left|\begin{array}{rrrrcr} 2x_1 & -x_2 & & +3x_4 & = & 9 \\ 4x_1 & -2x_2 & +5x_3 & & = & -10 \\ 3x_1 & +5x_2 & +2x_3 & -3x_4 & = & 0 \\ & -x_2 & +x_3 & -x_4 & = & -7 \end{array}\right. .$$

Solution: We work the same way as with the Gauss method by choosing a pivot element from a row but the unknowns are excluded under the main diagonal as well as above it. The aim is to be left with non-zero elements only along the main diagonal.

$$\left(\begin{array}{rrrr|r} 2 & -1 & 0 & 3 & 9 \\ \boxed{4} & -2 & 5 & 0 & -10 \\ 3 & 5 & 2 & -3 & 0 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) \leftrightarrow \left(\begin{array}{rrrr|r} \boxed{4} & -2 & 5 & 0 & -10 \\ 2 & -1 & 0 & 3 & 9 \\ 3 & 5 & 2 & -3 & 0 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) :(4) \leftrightarrow$$

$$\left(\begin{array}{rrrr|r} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 2 & -1 & 0 & 3 & 9 \\ 3 & 5 & 2 & -3 & 0 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) .(-2)\ .(-3) \leftrightarrow \left(\begin{array}{rrrr|r} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & 0 & -\frac{5}{2} & 3 & 14 \\ 0 & \boxed{\frac{13}{2}} & -\frac{7}{4} & -3 & \frac{15}{2} \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) \leftrightarrow$$

$$\left(\begin{array}{rrrr|r} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & \boxed{13/2} & -\frac{7}{4} & -3 & \frac{15}{2} \\ 0 & 0 & -\frac{5}{2} & -3 & 14 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) :\left(\frac{13}{2}\right) \leftrightarrow \left(\begin{array}{rrrr|r} 1 & -\frac{1}{2} & \frac{5}{4} & 0 & -\frac{5}{2} \\ 0 & 1 & -\frac{7}{26} & -\frac{6}{13} & \frac{15}{13} \\ 0 & 0 & -\frac{5}{2} & -3 & 14 \\ 0 & -1 & 1 & -1 & -7 \end{array}\right) .\left(\frac{1}{2}\right)\ .(1) \leftrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{29}{26} & -\frac{3}{13} & -\frac{25}{13} \\ 0 & 1 & -\frac{7}{26} & -\frac{6}{13} & \frac{15}{13} \\ 0 & 0 & \boxed{-5/2} & 3 & 14 \\ 0 & 0 & \frac{19}{26} & -\frac{19}{13} & -\frac{76}{13} \end{array}\right) \quad :(-\tfrac{5}{2}) \quad \leftrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & \frac{29}{26} & -\frac{3}{13} & -\frac{25}{13} \\ 0 & 1 & -\frac{7}{26} & -\frac{6}{13} & \frac{15}{13} \\ 0 & 0 & 1 & -\frac{6}{5} & -\frac{28}{5} \\ 0 & 0 & \frac{19}{26} & -\frac{19}{13} & -\frac{76}{13} \end{array}\right) \quad .(\tfrac{7}{26}) \quad .(-\tfrac{29}{26}) \quad .(-\tfrac{19}{26}) \quad \leftrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{72}{65} & \frac{281}{65} \\ 0 & 1 & 0 & -\frac{51}{65} & -\frac{23}{65} \\ 0 & 0 & 1 & -\frac{6}{5} & -\frac{28}{5} \\ 0 & 0 & 0 & \boxed{-\frac{38}{65}} & -\frac{114}{65} \end{array}\right) \quad \leftrightarrow \quad :(-\tfrac{38}{65})$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{72}{65} & \frac{281}{65} \\ 0 & 1 & 0 & -\frac{51}{65} & -\frac{23}{65} \\ 0 & 0 & 1 & -\frac{6}{5} & -\frac{28}{5} \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \quad .(\tfrac{6}{5}) \quad .(\tfrac{51}{65}) \quad .(-\tfrac{72}{65}) \quad \leftrightarrow \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \quad \leftrightarrow \quad \boxed{\begin{array}{lr} x_1 = & 1 \\ x_2 = & 2 \\ x_3 = & -2 \\ x_4 = & 3 \end{array}}$$

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